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# INVARIANT MEANS ON FUNCTION SPACES ON SEMITOPOLOGICAL SEMIGROUPS

by

BRIAN EDMUND FORREST

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ................................
..............INVARIANT MEANS ON FUNCTION SPACES..............
..................ON SEMITOPOLOGICAL SEMIGROUPS..................
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in partial fulfilment of the requirements for the degree of Master of Science.

DEDICATION

To Steve, Ron, Byron and Trevor

ABSTRACT

A semigroup $S$ together with a Hausdorff topology $\mathcal{T}$ which makes multiplication separately continuous is called a semitopological semigroup. If $F$ is a right translation invariant subspace of continuous bounded complex-valued functions on $S$, then a right invariant mean on $F$ is a positive element of norm 1 in the dual $F^*$ of $F$ which is also invariant with respect to right translation of elements in $F$. In this thesis we examine conditions for and consequences of the existence of right invariant means on certain special subspaces of continuous bounded functions.

The first half of the thesis is concerned with a number of geometric properties which stem from the existence of right invariant means. In particular, we prove a generalization of a Hahn-Banach type extension theorem due to Silverman and give a new proof of a geometric property of Glicksberg. We exhibit a number of conditions equivalent to the existence of right invariant means and apply these results to the representation of groups.

The second half of the thesis demonstrates the importance of the structure of right ideals of a semigroup in determining whether certain function spaces will admit right invariant means. We also look at right thick subsets and, using a result of Mitchell's establish a condition under which a semigroup will contain a right ideal which is also a group.

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# CHAPTER 1

## INTRODUCTION

A semigroup $S$ together with a Hausdorff topology $\mathcal{T}$ is called a semitopological semigroup if for each $t \in S$, the maps $s \mapsto ts$ and $s \mapsto st$ are continuous. Let $F$ be a norm closed, conjugate closed linear subspace of continuous, bounded, complex-valued functions on $S$, which contains the constant functions. A positive element of norm one in the dual $F^*$ of $F$ is called a mean on $F$. If $F$ is invariant under right translation by elements of $S$, and if $\mu$ is a mean on $F$, then $\mu$ is called a right invariant mean if it is also invariant with respect to right translation. In this thesis, we investigate some of the properties of semitopological semigroups with function spaces that admit right invariant means.

The second chapter contains a summary of the more important definitions and notation used throughout the thesis. A number of well known results are stated without proof.

We begin the third chapter with a generalization of a result of Silverman [31, p. 235] relating invariant means and certain Hahn-Banach extension properties. These results are then applied to representations of locally compact groups.

The remainder of the chapter is devoted to characterizations of semitopological semigroups with function spaces which admit multiplicative right invariant means and an interesting geometric property of Glicksberg [10, p. 99].

In the fourth chapter we examine the important role played by right ideals in determining whether certain function spaces will admit

right invariant means. We also look at right thick subsets and use a result of Mitchell (Lemma 4.3.3) to demonstrate a condition under which a semigroup with a finite right thick subset will also contain a right ideal which is a group.

The last section of each chapter consists of examples which illustrate concepts developed earlier.

CHAPTER 2

PRELIMINARIES

## 2.1 Definitions and Notation

Let $S$ be a semigroup with a Hausdorff topology $\mathcal{T}$. Multiplication in $S$ is said to be *separately continuous* [resp. *jointly continuous*] if the maps $s \mapsto ts$ and $s \mapsto st$ are continuous for each $t \in S$ [resp. the map $(s,t) \mapsto st$ is continuous]. A semigroup $S$ with separately [resp. jointly] continuous multiplcation is called a *semitopological* [resp. *topological*] *semigroup*. If $\mathcal{T}$ is discrete then $S$ is called a *discrete semigroup*.

*CB(S)* will denote the Banach space of all bounded continuous complex valued functions on $S$. In the case that $S$ is discrete, $CB(S)$ is denoted by $\ell_\infty(S)$.

For each $s \in S$ we define the *right* [resp. *left*] *translation operator,* $R_s$, [resp. $L_s$] on $CB(s)$ by

$$(R_sf)(t) = f(ts) \quad [\text{resp.} \quad (L_sf)(t) = f(st)]$$

The set $O_R(f) = \{R_sf;\ s \in S\}$ [resp. $O_Lf = \{L_sf;\ s \in S\}$] is called the *right* [resp. *left*] *orbit of* $f$.

A linear subspace $F \subseteq CB(S)$ is *right* [resp. *left*] *invariant* if $O_R(f) \subseteq F$ [resp. $O_L(f) \subseteq F$] for each $f \in F$. $F$ is *invariant* if it is both left and right invariant.

If $F$ is a norm closed, conjugate closed linear subspace of $CB(S)$ which contains the constant functions then a *mean* on $F$ will be any element $\mu$ of the dual $F^*$ of $F$ for which $\mu(X_S) = 1 = ||\mu||$ where $X_A$ is the characteristic function of $A$ for $A \subseteq S$. The set

of all means on $F$ is denoted by $M(F)$.

Each $s \in S$ induces a mean $p_s$ on $F$ defined by

$$p_s(f) = f(s) \quad \text{for each} \quad f \in F$$

A *finite mean* is a mean $\mu$ of the form

$$\mu = \sum_{i=1}^{n} \alpha_i p_{s_i} \quad \text{where} \quad s_i \in S, \quad \alpha_i > 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

REMARK 2.1.1. $M(F)$ is a convex, weak-$*$ compact subset of the unit ball of $F^*$. An application of the Hahn-Banach theorem shows that the set of finite means is weak-$*$ dense in $M(F)$ (see [5]).

DEFINITION 2.1.2. Let $F$ be right [resp. left] invariant. A mean $\mu$ on $F$ is *right* [resp. *left*] *invariant* if $\mu(f) = \mu(R_s f)$ [resp. $\mu(f) = \mu(L_s f)$] for each $s \in S$. If in addition $F$ is a subalgebra, then a mean $\mu$ on $F$ is called *multiplicative* if $\mu(fg) = \mu(f)\mu(g)$ for each $f, g \in F$.

REMARK 2.1.3. Let $S$ be a semitopological semigroup and $F$ a right invariant norm closed, conjugate closed subspace of $CB(S)$ which contains the constants. Let

$$H = \{h \in F;\ h = \sum_{k=1}^{n} (f_k - R_{s_k} f_k),\ f_k \in F;\ s_k \in S\}.$$

A well known criterion of Dixmier states that $F$ has a right invariant mean if and only if $\sup\{\mathrm{Re}\,h(s); s \in S\} \geq 0$ for each $h \in H$ (see [16, p. 231]).

## 2.2 Subspaces of CB(S)

In this section we introduce the important subspaces of CB(S) which will be dealt with throughout this thesis. In addition, we will present some of the elementary properties of the spaces of almost periodic and weakly almost periodic functions on S. For details of proofs, we refer the reader to Burkel's excellent survey [3].

DEFINITION 2.2.1. A function $f \in CB(S)$ is said to be *almost periodic* [resp. *weakly almost periodic*] if $O_R(f)$ is relatively compact in the norm [resp. weak] topology of CB(S). Denote the class of almost periodic [resp. weakly almost periodic] functions by AP(S) [resp. WAP(S)].

A function $f \in CB(S)$ is said to be *right uniformly continuous* [resp. *weakly right uniformly continuous*] if the map $s \to R_s f$ is continuous into the norm [resp. weak] topology of CB(S), for each $s \in S$. Denote the class of all right uniformly continuous [resp. weakly right uniformly continuous] functions on S by RUC(S) [resp. WRUC(S)].

As is well known, AP(S), WAP(S), RUC(S), and WRUC(S) are right invariant, norm closed, conjugate closed subalgebras of CB(S) which contain the constant functions. Furthermore, $AP(S) \subseteq WAP(S) \subseteq WRUC(S)$ and $AP(S) \subseteq RUC(S) \subseteq WRUC(S)$ (see [2, p. 125]).

In [8], deLeuuw and Glicksberg established the following result:

THEOREM 2.2.2 [deLeuuw and Glicksberg]. Let S be a semitopological semigroup and $f \in CB(S)$. Then

i) If $S$ is compact, $O_R(f)$ is weakly compact.

ii) If $S$ is a compact topological semigroup then $O_R(f)$ is norm compact.

As a consequence of the above theorem, we see that if $S$ is a compact topological semigroup, then $AP(S) = WAP(S) = CB(S)$.

REMARK 2.2.3. Let $B(AP(S))$ [resp. $B(WAP(S))$] denote the space of all continuous linear operators on $AP(S)$ [resp. $WAP(S)$]. It is well known that the closure, $S^a$ [resp. $S^w$] of $\{R_s;\ s \in S\}$ in $B(AP(S))$ [resp. $B(WAP(S)$] with respect to the strong [resp. weak] operator topology is a compact topological [resp. semitopological] semigroup. $S^a$ [resp. $S^w$] is called the *almost periodic* [resp. *weakly almost periodic*] *compactification of* $S$ (see [3, p. 4]). With the aid of Mazur's theorem and the existence of $S^a$, the following characterization of $AP(S)$ can be deduced.

THEOREM 2.2.4. Let $f \in CB(S)$. Then the following are equivalent:

(i) $f \in AP(S)$.

(ii) $O_L(f)$ is relatively compact in the norm topology of $CB(S)$.

(iii) The convex hull, $coO_R(f)$, of $O_R(f)$ is relatively compact in the norm topology of $CB(S)$.

(iv) $coO_L(f)$ is relatively compact in the norm topology of $CB(S)$.

For $WAP(S)$ we have

THEOREM 2.2.5. Let $f \in CB(S)$. Then the following are equivalent:

(i) $f \in WAP(S)$.

(ii) $O_L(f)$ is relatively compact in the weak topology of $CB(S)$.

(iii) $\mathrm{co}O_R(f)$ is relatively compact in the weak topology of $CB(S)$.

(iv) $\mathrm{co}O_L(f)$ is relatively compact in the weak topology of $CB(S)$.

(v) $\lim_n \lim_m f(s_n t_m) = \lim_m \lim_n f(s_n t_m)$ whenever $\{s_n\}$, $\{t_m\}$ are sequences in $S$ such that all of the limits exist.

REMARK 2.2.6. The equivalence of (i) and (v) in theorem 2.2.5 was established by Grothendieck in [14].

In [32], Von Neumann established the existence of a unique right invariant mean on $AP(G)$ for any semitopological semigroup $G$ which is algebraically a group. Using his fixed point theorem, Ryll-Nardzewski [28] showed that if $G$ is a group and a semitopological semigroup, then $WAP(G)$ also has a unique right invariant mean.

REMARK 2.2.7. There are many proofs of Ryll-Nardzewski's fixed point theorem. The original proof, using the Martingale Convergence Theorem, and a geometric proof of Namioka and Apslund (see [13, pp. 97-99]) are both well known. We wish to refer the reader to an interesting proof of Hansel and Troallic [15] which utilizes Hahn's theorem and a recent proof given by Namioka [25].

We close this section by stating a useful theorem of Granierer and Lau [12, p. 252]. We will need the following definitions.

DEFINITION 2.2.8. A right [resp. left] invariant, norm closed, conjugate closed subspace $F$ of $CB(S)$ which contains the constant functions is called *right* [resp. *left*] *introverted* if, for each $\mu \in M(F)$ and $f \in F$, the function $T_\mu(f)$, defined by

$$(T_\mu f)(s) = \mu(R_s f) \quad [\text{resp. } (T_\mu f)(s) = \mu(L_s f)],$$

is in $F$. $F$ is *introverted* if it is both left and right introverted.

THEOREM 2.2.9 [Granierer and Lau]. Let $S$ be a semitopological semigroup and let $F$ be an introverted subspace of $CB(S)$. Let $\mu = \sum_{i=1}^{n} \alpha_i p_{s_i}$ be a finite mean. Define $\ell_\mu(f) = \sum_{i=1}^{n} \alpha_i L_{s_i} f$ for each $f \in F$. If $F$ has a right invariant mean, then for any $f \in F$, $\{\phi(f);\ \phi$ is a right invariant mean on $F\} = \{c;\ cX_s \in$ pointwise closure of $\{\ell_\mu(f);\ \mu$ is a finite mean$\}\}$.

## 2.3 Examples

EXAMPLE 2.3.1. Let $\mathbb{R}$ denote the additive group of real numbers with the usual topology. Then $\mathbb{R}$ is a topological semigroup.

A subset $A$ of $\mathbb{R}$ is *relatively dense* if there exists an $r > 0$ such that every interval of length $r$ meets $A$. Bohr defined a function $f \in CB(\mathbb{R})$ to be almost periodic if for each $\epsilon > 0$ there exists a relatively dense subset $E(\epsilon)$ of $\mathbb{R}$ such that

$$\|f(x+z) - f(x)\| < \epsilon \quad \text{for each } z \in E(\epsilon).$$

The equivalence of Bohr's definition of $AP(\mathbb{R})$ with that of definition 2.2.1 was shown by Bochner (see [1,p. 7]). It was in the spirit of Bochner's result that Eberlein first introduced the space $WAP(S)$ in [9].

Bochner's famous Approximation Theorem (see [1]) showed that $AP(\mathbb{R})$ is in fact the closure in the norm topology of the set $\{P(t);\ P(t) = \sum_{j=1}^{n} e^{i\lambda_j t}\}$ of trigonometric polynomials.

The right invariant mean $\phi$ on $WAP(\mathbb{R})$ is given by

$$\phi(f) = \lim_{t\to\infty} \int_{-t}^{t} f(t)dt \quad \text{for each} \quad f \in WAP(\mathbb{R})$$

For further details on $AP(\mathbb{R})$ and $WAP(\mathbb{R})$ we refer the reader to [1].

EXAMPLE 2.3.2. Let $S = \{A,B\}$ with discrete topology. Define "$\circ$" on $S$ by

$$1)\ A \circ A = A \circ B = A \qquad 2)\ B \circ B = B \circ A = B.$$

By theorem 2.2.2 $AP(S) = WAP(S) = \ell_\infty(S)$. Consider the function $f = X_{\{A\}}$. Then $(L_A f)(A) = (L_A f)(B) = 1$ while $(L_B f)(A) = (L_B f)(B) = 0$. Therefore, $AP(S)$ does not have a left invariant mean. However, for each $0 \leq \alpha \leq 1$ the mean $\phi_\alpha$ defined by

$$\phi_\alpha(f) = \alpha f(A) + (1-\alpha)f(B) \quad \text{for each} \quad f \in \ell_\infty(S)$$

is a right invariant mean on $\ell_\infty(S)$.

EXAMPLE 2.3.3. Let $S$ be a discrete semigroup. Let $x$ be a left zero of $S$, that is $xy = x$ for each $y \in S$. Then $p_x$ is a right invariant mean on $\ell_\infty(S)$. Furthermore, $p_x$ is multiplicative.

EXAMPLE 2.3.4. Let $G$ be the free group with two generators and with the discrete topology. Then $\ell_\infty(G)$ does not have a right invariant mean (see [16, p. 236]), while $WAP(G)$ does have a right invariant mean.

If $S = G \cup \{z\}$ is the one point compactification of $G$ with

$xz = zx = z$ for each $x \in S$, then $S$ is a compact semitopological semigroup with a dense subgroup $G$ for which $CB(S)$ has a multiplicative right invariant mean while $CB(G)$ does not have a right invariant mean.

Contrast this with a result of Hoffman and Mosert [17], which states that if $S$ is a topological semigroup with a dense subgroup $G$, then $S$ is algebraically a group.

# CHAPTER 3

# HAHN-BANACH EXTENSIONS AND PROPERTIES OF RIGHT INVARIANT MEANS

## 3.1 Introduction

In this chapter we examine some of the consequences of the existence of right invariant means on subspaces of $CB(S)$.

We begin with a generalization of a theorem of Silverman's [31, p. 235] dealing with the extension of translation invariant linear functionals, and then apply the result to certain representation of locally compact groups.

In Section 3.3, we give a characterization of those groups $G$ for which $AP(G)$ and $WAP(G)$ admit multiplicative right invariant means, as the class of groups for which $AP(G)$ consists of constant functions only.

In Section 3.4, we present a new proof of an interesting geometric property of Glicksberg possessed by semitopological semigroups for which $CB(S)$ has a right invariant mean. We also mention an analog of Glicksberg's result for $WAP(S)$.

The final section consists of examples illustrating concepts developed earlier in the chapter.

## 3.2 Hahn-Banach Extension Properties

DEFINITION 3.2.1. Let $S$ be a semitopological semigroup. A *representation of* $S$ is a homomorphism $\rho$ of $S$ onto a semigroup $\{T_s; s \in S\}$ of linear operators on a vector space $X$. We will denote

a representation $\rho$ by its image set $\{T_s;\ s \in S\}$.

A representation $\{T_s;\ s \in S\}$ of $S$ on a topological vector space $X$ is said to be *weakly continuous* if the map $s \to T_s x$ is continuous into the weak topology of $X$, for each $x \in X$.

A representation $\{T_s;\ s \in S\}$ of $S$ on a Hilbert space $\mathcal{H}$ is said to be *unitary* if the operator $T_s$ is unitary for each $s \in S$.

DEFINITION 3.2.2. Let $S$ be a semitopological semigroup and let $\mathcal{F}$ be an invariant, norm closed, conjugate closed subspace of $CB(S)$ which contains the constant functions.

$S$ is said to have the *1st Hahn-Banach Extension Property* (HBEP1), with respect to $\mathcal{F}$ if given any representation $\{T_s;\ s \in S\}$ of $S$ as a semigroup of linear operators on a vector space $X$, a semi-norm $p$ on $X$, a linear subspace $Y$ of $X$ and a linear functional $\phi$ on $Y$ which satisfies (i) to (v) below

(i) $T_s(Y) \subseteq Y$ for each $s \in S$

(ii) $p(T_s x) \leq p(x)$ for each $s \in S$, $x \in X$

(iii) $|\phi(y)| \leq p(y)$ for each $y \in Y$

(iv) $\phi(T_s y) = \phi(y)$ for each $s \in S$, $y \in Y$

(v) $\phi$ has an extension to a linear functional $\psi$ on $X$ such that $|\psi(x)| \leq p(x)$ and the function $\psi_x\colon S \to \mathbb{C}$ defined by $\psi_x(S) = \psi(T_s x)$ is in $\mathcal{F}$ for each $x \in X$,

then $\phi$ must have an extension to a linear functional $\Phi$ on $X$ with

(a) $|\Phi(x)| \leq p(x)$ for each $x \in X$.

(b) $\Phi(T_s x) = \Phi(x)$ for each $s \in S$, $x \in X$.

$S$ is said to have the *2nd Hahn-Banach Extension Property* (HBEP2), with respect to $F$ if given any representation $\{T_s;\ s \in S\}$ of $S$ as a semigroup of linear operators on a normed linear space $X$, a closed subspace $Y$ of $X$ and $\phi \in Y^*$ satisfying (i) to (iv) below

(i) $\|T_s\| \leq 1$ for each $s \in S$

(ii) $T_s(Y) \subseteq Y$ for each $s \in S$, $y \in Y$

(iii) $\phi(T_s y) = \phi(y)$ for each $s \in S$, $y \in Y$

(iv) $\phi$ has an extension $\psi \in X^*$ with $\|\phi\| = \|\psi\|$ such that the function $\psi_x \in F$ for each $x \in X$.

then $\phi$ must have an extension $\Phi \in X^*$ such that

(a) $\|\Phi\| = \|\phi\|$

(b) $\Phi(T_s x) = \Phi(x)$ for each $s \in S$, $x \in X$.

In [31], Silverman showed that a discrete semigroup $S$ will possess the (HBEP1) with respect to $\ell_\infty(S)$ if and only if $\ell_\infty(S)$ admits a right invariant mean. The next theorem is a generalization of Silverman's result.

THEOREM 3.2.3. Let $S$ be a semitopological semigroup and let $F$ be an invariant, norm closed, conjugate closed subspace of $CB(S)$ which contains the constant functions. Then the following are equivalent:

(i) $S$ has the (HBEP1) with respect to $F$.

(ii) $S$ has the (HBEP2) with respect to $F$.

(iii) $F$ has a right invariant mean.

PROOF.

(i) implies (ii)

Let $\{T_s;\ s \in S\}$ be a representation of $S$ as a semigroup of linear operators on a normed linear space $X$ such that $\|T_s\| \leq 1$ for each $s \in S$. Let $Y$, $\phi$ and $\psi$ be as in the definition of (HBEP2). Define the seminorm $p$ on $X$ by $p(x) = \|\phi\|\,\|x\|$. Since $\|T_s\| \leq 1$, $p(T_s x) \leq p(x)$, for each $s \in S$, $x \in X$. Clearly, $|\phi(y)| \leq p(y)$ for each $y \in Y$, while $\|\psi\| = \|\phi\|$ implies that $|\psi(x)| \leq p(x)$ for each $x \in X$. Therefore, the conditions of the (HBEP1) are satisfied for the representation $\{T_s;\ s \in S\}$.

Since $S$ has the (HBEP1) with respect to $F$, $\phi$ has an extension $\Phi$ on $X$ such that $|\Phi(x)| \leq p(x)$ and $\Phi(T_s x) = \Phi(x)$ for each $s \in S$, $x \in X$. But then $|\Phi(x)| \leq \|\phi\|\,\|x\|$, and since $\Phi$ extends $\phi$, $\|\Phi\| = \|\phi\|$. It follows that $S$ has the (HBEP2) with respect to $F$.

(ii) implies (iii)

Suppose that $S$ has the (HBEP2) with respect to $F$. Let $X = F$, $Y = \{\alpha X_S;\ \alpha \in \mathbb{C}\}$ and let $\{R_s;\ s \in S\}$ be the right regular representation of $S$ on $F$. Define the linear functional $\phi$ on $Y$ by

$$\phi(\alpha X_S) = \alpha \quad \text{for each} \quad \alpha \in \mathbb{C}$$

Let $\psi$ be the linear functional defined on $F$ by

$$\psi(f) = f(s_o) \quad \text{for each} \quad f \in F$$

where $s_o$ is any fixed element of $S$. Then $\psi$ extends $\phi$ to $F$.

Observe that for each $t \in S$, $f \in F$ we have:

$$\psi_f(t) = \psi(R_t f) = R_t f(s_o) = f(s_o t) = L_{s_o} f(t)$$

But $F$ is left invariant, so $\psi_f \in F$. It is clear tht the conditions (i) to (iv) of the (HBEP2) have been satisfied. Therefore, $\phi$ has an extension to a linear functional $\Phi \in F^*$ such that $||\Phi|| = ||\phi|| = 1$ and $\Phi(R_s f) = \Phi(f)$ for each $s \in S$, $f \in F$. Furthermore, since $\Phi$ extends $\phi$, $\Phi(X_s) = 1$ and $\Phi$ is a right invariant mean on $F$.

(iii) implies (i)

Let $\mu$ be a right invariant mean on $F$. Let $\{T_s; s \in S\}$ be a representation of $S$ as a semigroup of linear operators on a vector space $X$. Let $Y$ be a subspace of $X$, $p$ a seminorm on $X$, $\phi$ and $\psi$ linear functionals on $Y$ and $X$ respectively such that conditions (i) to (v) of the (HBEP1) are satisfied. Define $\Phi$ on $X$ by

$$\Phi(x) = \mu(\psi_x) \quad \text{for each} \quad x \in X.$$

We first show that $\Phi$ is linear.

Let $x, y \in X$, $\alpha, \beta \in \mathbb{C}$, then $\psi_{\alpha x+\beta y} = \alpha\psi_x + \beta\psi_y$ by the linearity of $\psi$ and of each $T_s$. Therefore,

$$\Phi(\alpha x+\beta y) = \mu(\psi_{\alpha x+\beta y}) = \alpha\mu(\psi_x) + \beta\mu(\psi_y) = \alpha\Phi(x) + \beta\Phi(y)$$

and $\Phi$ is linear.

Note that $\psi_y(s) = \phi(T_s y) = \phi(y)$ for each $y \in Y$, $s \in S$ since

$\psi$ is an extension of the $\{T_s;\ s \in S\}$-invariant linear functional $\phi$. That is $\psi_y = \phi(y)\cdot X_S$ for each $y \in Y$. Since $\mu$ is a mean,

$$\Phi(y) = \mu(\psi_y) = \mu(\phi(y)\cdot X_S) = \phi(y) \quad \text{for each } y \in Y,$$

so $\Phi$ is an extension of $\phi$.

Observe that

$$\psi_{T_s x}(t) = \psi(T_t(T_s x)) = \psi(T_{ts}x) = \psi_x(ts) = R_s\psi_x(t)$$

for each $s, t \in S$, $x \in X$. It follows that

$$\Phi(T_s x) = \mu(\psi_{T_s x}) = \mu(R_s\psi_x) = \mu(\psi_x) = \Phi(x)$$

for each $s \in S$, $x \in X$.

Since $\mu$ is a mean, $|\mu(\psi_x)| \leq \sup_{s\in S} |\psi_x(s)|$. From this we see that

$$|\Phi(x)| = |\mu(\psi_x)| \leq \sup_{s\in S} |\psi_x(s)| = \sup_{s\in S} |\psi(T_s x)| \leq \sup_{s\in S} p(T_s x) \leq p(x)$$

for each $x \in X$. Therefore, $S$ has the (HBEP1) with respect to $\mathcal{F}$.

REMARK 3.2.4. In the proof that the (HBEP2) implies the existence of a right invariant mean, we defined $\psi$ by $\psi(f) = f(s_o)$ for some fixed $s_o \in S$. If $S$ has an identity $e$, and we let $s_o = e$, then $\psi_f = f$ and the assumption that $\mathcal{F}$ be left invariant may be dropped.

REMARK 3.2.5. Let $\{T_s;\ s \in S\}$ be a representation of a semitopological semigroup $S$ as a semigroup of linear operators on a normed linear space $X$ such that $||T_s|| \leq M$ for each $s \in S$. If we replace

condition (a) in the definition of the (HBEP2) with

(a') $$\|\Phi\| \leq M\|\phi\|,$$

then $S$ will have this extension property with respect to $F$ whenever $F$ has a right invariant mean.

The next lemma, originally stated for commutative semigroups, was first proved by Eberlein in [9]. A simple proof can be found in [3, p. 36].

LEMMA 3.2.6. Let $S$ be a semitopological semigroup and let $\{T_s; s \in S\}$ be a weakly continuous representation of $S$ as a uniformly bounded semigroup of linear operators on a Banach space $X$. If for some $x \in X$ the set $O_x = \{T_s x; s \in S\}$ is relatively compact [resp. weakly relatively compact], then the function $f_x$ defined by

$$f_x(s) = \phi(T_s x) \quad \text{for each} \quad s \in S$$

is in $AP(S)$ [resp. $WAP(S)$] for each $\phi \in X^*$.

DEFINITION 3.2.7. A topological semigroup $G$ which is algebraically a group is called a *topological group* if the map $x \mapsto x^{-1}$ is continuous. A *locally compact group* is a topological group with a locally compact topology.

THEOREM 3.2.8. Let $G$ be a locally compact group and let $\{T_g; g \in G\}$ be a weakly continuous representation of $G$ as a group of linear operators on a Banach space $X$ such that $\|T_g\| \leq 1$ for each $g \in G$. Suppose that $O_x = \{T_g x; g \in G\}$ is relatively weakly compact for each $x \in X$. Let $Y$ be a closed subspace of $X$, invariant under $\{T_g; g \in G\}$,

and $\phi \in Y^*$ be such that $\phi(T_s y) = \phi(y)$ for each $g \in G$, $y \in Y$. Then $\phi$ has an extension $\Phi \in X^*$ such that $||\Phi|| = ||\phi||$ and $\Phi(T_g x) = \Phi(x)$ for each $g \in G$, $x \in X$.

PROOF. Recall, since $G$ is a locally compact group, $WAP(G)$ has a right invariant mean. By Theorem 3.2.3, $G$ has the (HBEP2) with respect to $WAP(G)$.

By the usual Hahn-Banach theorem, $\phi$ has an extension $\psi \in X^*$ with $||\psi|| = ||\phi||$. Since $O_x$ is relatively weakly compact, $\psi_x$ is in $WAP(G)$ for each $x \in X$ (Lemma 3.2.6). Condition (iv) of the (HBEP2) is satisfied and the result follows immediately.

COROLLARY 3.2.9. Let $G$ be a locally compact group and let $\{T_g;\ g \in G\}$ be a weakly continuous representation of $G$ on a reflexive Banach space $X$ such that $||T_g|| \leq 1$ for each $g \in G$. Let $Y$ be a closed subspace of $X$ invariant under $\{T_g;\ g \in G\}$ and $\phi \in Y^*$ be such that $\phi(T_g y) = \phi(y)$ for each $g \in G$, $y \in Y$. Then $\phi$ has an extension $\Phi \in X^*$ with $||\Phi|| = ||\phi||$ and $\Phi(T_g x) = \Phi(x)$ for each $g \in G$, $x \in X$.

PROOF. If $X$ is reflexive, then the unit ball of $X$ is weakly compact. Therefore $O_x = \{T_g x;\ g \in G\}$ is relatively weakly compact for each $x \in X$. Now apply Theorem 3.2.8.

REMARK 3.2.10. Let $G$ be a locally compact group and $\{T_g;\ g \in G\}$ be a weakly continuous representation of $G$ on a Banach space $X$ such that $||T_g|| \leq 1$ for each $g \in G$ and $\{T_g\ ;\ g \in G\}$ has a non-zero fixed point $y$, then the function $\phi(\alpha y) = \alpha||y||$ is a continuous

invariant linear functional on the 1-dimensional subspace $Y = \{\alpha y;\ \alpha \in \mathbb{C}\}$. Theorem 3.2.8 implies that $\phi$ has an extension $\Phi \in X^*$ such that $||\Phi|| = ||\phi|| = 1$ and $\Phi(T_g x) = \Phi(x)$ for each $g \in G$, $x \in X$. Therefore, the existence of a non-zero fixed point implies the existence of a non-zero $\{T_g;\ g \in G\}$-invariant linear functional on $X$.

In the case where $X$ is a Hilbert space and $\{U_g;\ g \in G\}$ is a weakly continuous unitary representation on $X$, then the converse of the above remark holds. Indeed, if $\phi$ is a non-zero continuous $\{U_g;\ g \in G\}$-invariant linear functional on $X$, then there exists a non-zero $z \in X$ such that $\phi(x) = \langle x,z\rangle = \langle U_g x,z\rangle = \phi(U_g x)$ for each $g \in G$, $x \in X$. If $z$ is not a fixed point of $\{U_g;\ g \in G\}$, then for some $g_1, g_2 \in G$, $U_{g_1} z \neq U_{g_2} z$. But then there exists $y \in X$ such that $\langle y, U_{g_1} z\rangle \neq \langle y, U_{g_2} z\rangle$. However, this would imply that $\phi(U_{g_1}^{-1} y) = \langle U_{g_1}^{-1} y, z\rangle = \langle y, U_{g_1} z\rangle \neq \langle y, U_{g_2} z\rangle = \langle U_{g_2}^{-1} y, z\rangle = \phi(U_{g_2}^{-1} y)$ which is impossible since $\phi$ is $\{U_g;\ g \in G\}$-invariant. It follows that $z$ is a fixed point of $\{U_g;\ g \in G\}$.

## 3.3 Multiplicative Right Invariant Means and Groups

Let $G$ be a locally compact group. In [12], Granierer and Lau proved that if $G$ is non-trivial, then $RUC(G)$ does not have a multiplicative right invariant mean. In this section we characterize these groups $G$ for which $WAP(G)$ admits a multiplicative right invariant mean.

Our main lemma can be found as a remark in [2, p. 164]. The proof below was suggested by A.T. Lau. It is much shorter than our original proof.

LEMMA 3.3.1. Let $G$ be a group and a semitopological semigroup. Then $AP(G)$ has a multiplicative right invariant mean, if and only if $AP(G)$ consists of constant functions only.

PROOF. If $AP(G)$ consists only of constant functions, then $\mu \in AP(G)^*$ defined by $\mu(\alpha X_s) = \alpha$ for each $\alpha \in \mathbb{C}$, is clearly a multiplicative right invariant mean on $AP(G)$.

Conversely, if $AP(G)$ has a multiplicative right invariant mean and if $f \in AP(G)$, then, by a result of Granierer and Lau [12], there exists a net $\{g_\alpha\}_{\alpha \in A}$ in $G$ such that $L_{g_\alpha} f \to cX_s$ for some $c \in \mathbb{C}$, in the pointwise topology on $CB(G)$. Since the left orbit of $f$ is relatively compact, $L_{g_\alpha} f \to cX_s$ in the norm topology of $CB(S)$. Hence, for each $\epsilon > 0$ there exists $\alpha_0 \in A$ such that if $\alpha > \alpha_0$, then

$$|f(g_\alpha t) - c| < \epsilon \quad \text{for each} \quad t \in G.$$

But since $G$ is a group

$$|f(t) - c| < \epsilon \quad \text{for each} \quad t \in G.$$

Since $\epsilon > 0$ was arbitrary, $f(t) = c$ for each $t \in G$.

COROLLARY 3.3.2. Let $G$ be a locally compact abelian group. If $G$ is non-trivial, then $AP(G)$ does not have a multiplicative right invariant mean.

PROOF. If $G$ is non-trivial, then since the continuous characters separate points, and are contained in $AP(G)$, $AP(G)$ must contain non-constant functions. By Lemma 3.3.1, $AP(G)$ does not have a

multiplicative right invariant mean.

REMARK 3.3.3. A locally compact group $G$ is called *minimally almost periodic* if $AP(G)$ consists only of constant functions. In [4] Chou, utilizing the Iwasawa decomposition (see [30]), established the minimal almost periodicity of the group $SL(2,\mathbb{R})$.

A well known result of Segal and von Neumann [29, p. 515] states that any non-compact simple Lie group is minimally almost periodic.

A locally compact group $G$ is called *maximally almost periodic* if the finite dimensional irreducible representations of $G$ separate points. If $G$ is a non-trivial maximally almost periodic group, then by Lemma 3.3.1, $AP(G)$ does not have a multiplicative right invariant mean. Every locally compact abelian group is maximally almost periodic, so the above remark generalizes Corollary 3.3.2. For details about the class of maximally almost periodic groups we refer the reader to an excellent survey by Palmer [26] on the classes of locally compact groups.

COROLLARY 3.3.4. Let $G$ be a non-trivial locally compact group which is continuously isomorphic to a subgroup of a compact group. Then $AP(G)$ does not have a multiplicative right invariant mean.

PROOF. Every locally compact group $G$ which is continuously isomorphic to a subgroup of a compact group is maximally almost periodic (see [26, p. 692]).

The next lemma, which is due to Burckel [3, p. 30], will be used to prove our main result.

LEMMA 3.3.5. Let $G$ be a group and a semitopological semigroup. Let

$W_0(G) = \{f \in WAP(G): \mu(|f|) = 0\}$ where $\mu$ is the unique right invariant mean on $WAP(G)$. Then $WAP(G) = AP(G) \oplus W_0(G)$ is a direct sum decomposition of $WAP(G)$.

THEOREM 3.3.6. Let $G$ be a group and a semitopological semigroup. Then $WAP(G)$ has a multiplicative right invariant mean if and only if $AP(G)$ consists only of constant functions.

PROOF. If $WAP(G)$ has a multiplicative right invariant mean $\mu$, then the restriction of $\mu$ to $AP(G)$ is a multiplicative right invariant mean on $AP(G)$. By Lemma 3.3.1, $AP(G)$ consists only of constant functions.

Conversely, assume that $AP(G) = \{\alpha X_G; \alpha \in \mathbb{C}\}$. Let $\mu$ be the unique right invariant mean on $WAP(G)$. Let $WAP(G) = AP(G) \oplus W_0(G)$ be the direct sum decomposition of $WAP(G)$ induced by $\mu$. Let $f \in WAP(G)$. Then $f = \alpha X_G + h$ where $h \in W_0(G)$ and $\mu(f) = \alpha$.

Let $f, g \in WAP(G)$. If $f = \alpha_1 X_G + h_1$ and $g = \alpha_2 X_G + h_2$ where $h_1, h_2 \in W_0(G)$, then $fg = (\alpha_1\alpha_2)X_G + \alpha_2 h_1 + \alpha_1 h_2 + h_1 h_2$. By the linearity of $\mu$, $\mu(fg) = \alpha_1\alpha_2 = \mu(f)\mu(g)$, so $\mu$ is multiplicative. Since $\mu$ is right invariant, $WAP(G)$ has a multiplicative right invariant mean.

DEFINITION 3.3.7. Let $X$ be a non-empty topological space. A *right action* of a semigroup $S$ on $X$ is a map, $(x,s) \mapsto xs$, from $X \times S$ into $X$, such that $x(st) = (xs)t$ for each $x \in X$, $s, t \in S$.

A right action on a vector space $X$ is said to be affine if for each $s \in S$, the map $x \mapsto xs$ is affine.

Let $S$ be a semitopological semigroup and let $X$ be a compact

Hausdorff space. A right action of $S$ on $X$ is called equicontinuous if given $\mathcal{U}$, the unique uniformity determining the topology of $X$ (see [19, p. 197]), $y \in X$ and $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $(xs,ys) \in \mathcal{U}$ for each $s \in S$, $(x,y) \in V$.

In [20], Lau established the following result:

LEMMA 3.3.8 [Lau]. Let $S$ be a semitopological semigroup. If a right action of $S$ on a compact Hausdorff space $X$ is both separately continuous and equicontinuous, then the function $T_y f(s) = f(ys)$ is in $AP(S)$ for each $f \in CB(S)$, $y \in X$.

THEOREM 3.3.9 [Lau]. $AP(S)$ has a multiplicative right invariant mean if and only if every separately continuous and equicontinuous right action of $S$ on a compact Hausdorff space has a fixed point, $x \in X$.

For groups Lau's results have an interesting application.

THEOREM 3.3.10. Let $G$ be a group and a semitopological semigroup. Then $WAP(G)$ has a multiplicative right invariant mean if and only if every separately continuous and equicontinuous right action of $G$ on a compact Hausdorff space $X$ is such that $xs = xe$ for each $s \in S$, $x \in X$, where $e$ denotes the identity element of $G$.

PROOF. Suppose that every separately continuous and equicontinuous right action of $G$ on a compact Hausdorff space $X$ is such that $xs = xe$ for each $s \in G$, $x \in X$. Let $x_0 \in X$. Then $x_0 e$ is a fixed point for $G$. By Theorem 3.3.9, $AP(G)$ has a multiplicative right invariant mean. By Theorem 3.3.7 and Lemma 3.3.1, $WAP(G)$ also has a multiplicative right invariant mean.

Conversely, assume that WAP(G) has a multiplicative right invariant mean. Then by Theorem 3.3.7, AP(G) consists of constant functions only.

Suppose that a right action of $G$ on a compact Hausdorff space $X$, is both separately continuous and equicontinuous. Assume also that $xg_1 \neq xg_2$ for some $g_1, g_2 \in G$, $x \in X$. Since $CB(X)$ separates points, there exists $f \in CB(X)$ such that $f(xg_1) \neq f(xg_2)$. However, by Lemma 3.3.8, $T_x f$ is a non-constant function in AP(G). Since this is impossible, it follows that $xg = xe$ for each $g \in G$, $x \in X$.

COROLLARY 3.3.11. Let $G$ be a group and a semitopological semigroup. Suppose that AP(G) has a multiplicative right invariant mean. Let $f \in$ WAP(G). Then right translation is equicontinuous with respect to the weak topology on the closure of $co\mathcal{O}_R(f)$ if and only if $f$ is a constant function.

Granierer, in [12], gave a characterization of the class of discrete semigroups for which $\ell_\infty(S)$ admits a multiplicative left invariant mean in terms of a certain multiplicative extension property. We now present a generalization of the right analog of Granierer's result for subalgebras of $CB(S)$, where $S$ is a semitopological semigroup.

DEFINITION 3.3.12. Let $S$ be a semitopological semigroup and let $F$ be a norm closed conjugate closed, invariant subalgebra of $CB(S)$ which contains the constant functions. $S$ is said to have the *Multiplicative Extension Property*, (MEP), with respect to $F$ if, given any representation. $\{T_s; s \in S\}$ of $S$ as a semigroup of continuous algebra homomorphisms from a Banach algebra $B$ into $B$,

a norm closed subalgebra $A$ of $B$, and $\phi \in A^*$ a multiplicative linear functional satisfying:

(i) $T_s(A) \subseteq A$ for each $s \in S$

(ii) $\phi(T_s y) = \phi(y)$ for each $s \in S$, $y \in A$

(iii) $\phi$ has an extension to a multiplicative linear functional $\psi \in B^*$ such that $\|\phi\| = \|\psi\|$ and the function $\psi_x(s) = \psi(T_s x)$ is in $F$ for each $x \in B$.

Then $\phi$ must have an extension to a multiplicative linear functional $\Phi \in B^*$ such that

(a) $\|\Phi\| = \|\phi\|$

(b) $\Phi(T_s x) = \Phi(x)$ for each $s \in S$, $x \in S$.

THEOREM 3.3.13. Let $S$ be a semitopological semigroup and let $F$ be a norm closed, conjugate closed, invariant subalgebra of $CB(S)$ which contains the constant functions. Then $S$ has the (M.E.P.) with respect to $F$ if and only if $F$ has a multiplicative right invariant mean.

REMARK 3.3.14. A straight forward modification of the proof of Theorem 2.2.3 establishes Theorem 3.3.13. This proof, we believe, is more elementary than Granierer's proof in that it does not rely on any fixed point theorem or any result on homomorphisms of semigroups.

COROLLARY 3.3.15. Let $\{T_g; g \in G\}$ be a weakly continuous representation of a minimally almost periodic group $G$ as a group of continuous algebra homomorphisms of a Banach algebra $B$ onto $B$, such that $\|T_g\| \leq 1$ for each $g \in G$ and $O_x = \{T_g x; g \in G\}$ is relatively weakly

compact for each $g \in G$, $y \in A$. Let $A$ be a norm closed subalgebra of $B$ for which $T_g(A) \subseteq A$ for each $g \in G$. Let $\phi \in A^*$ be a multiplicative linear functional such that $\phi(T_g y) = \phi(y)$ for each $g \in G$, $y \in Y$. If $\phi$ has a multiplicative extension $\psi \in B^*$, with $\|\psi\| = \|\phi\|$, then $\phi$ has a multiplicative extension $\Phi \in B^*$ for which $\|\Phi\| = \|\phi\|$ and $\Phi(T_g x) = \Phi(x)$ for each $g \in G$, $x \in X$.

PROOF. Since $G$ is minimally almost periodic, WAP(G) has a multiplicative right invariant mean. By Theorem 3.3.13, G has the (M.E.P.) with respect to WAP(G). $O_x$ is relatively weakly compact for each $x \in X$, so, by Lemma 3.2.6 the function $\psi_x$ is in WAP(G). The conditions of the (M.E.P.) are satisfied and Theorem 3.3.13 insures the desired extension.

## 3.4 A Geometric Property of Glicksberg

In this section we give a new proof of a geometric property associated with those semigroups $S$ for which CB(S) has a right invariant mean. The theorem was first proved by Glicksberg in [10]. A second proof using Mitchell's fixed point theorem was given by Granierer (see [11, p. 59]).

THEOREM 3.4.1. Let $S$ be a semitopological semigroup and let $\mu$ be a right invariant mean on CB(S). Let $\{T_s; s \in S\}$ be a continuous representation of $S$ as a semigroup of linear operators on a normed linear space $X$, such that $\|T_s\| \leq 1$ for each $s \in S$. If $x \in X$ is such that $d(0, coO_x) = \sup\{\|z\| ; z \in coO_x\} = \alpha > 0$, then there exists $\phi \in X^*$ with $\|\phi\| = 1$, $\phi(x) = \alpha$, and $\phi(T_s y) = \phi(y)$ for each $s \in S$, $y \in X$.

PROOF. Suppose that $d(0, coO_x) = \alpha > 0$. By the Hahn-Banach theorem, there exists $\psi \in X^*$ such that $||\psi|| = 1$ and $|\psi(z)| \geq \alpha$ for each $z \in coO_x$.

For each $y \in X$, define $\psi_y: S \to \mathbb{C}$ by $\psi_y(s) = \psi(T_s y)$. Then $\psi_y \in CB(S)$. Define $\phi(y) = \mu(\psi_y)$ for each $y \in Y$. $\phi$ is linear and invariant under $\{T_s; s \in S\}$. Since $||\mu|| = 1$ and $||T_s|| \leq 1$ for each $s \in S$, $||\phi|| \leq ||\psi|| = 1$. However, $|\psi(z)| \geq \alpha$ for each $z \in coO_x$. Since $\mu$ is a mean $|\phi(z)| \geq \alpha$ for each $z \in coO_x$. Therefore $|\phi(z_0)| = \alpha$ for some $z_0 \in coO_x$. We may assume that $\phi(z_0) = \alpha$. Since $\phi$ is invariant, $\phi(x) = \alpha$.

COROLLARY 3.4.2 [Glicksberg]. Let $S$ be a semitopological semigroup and let $\{T_s; s \in S\}$ be a continuous representation of $S$ as a semigroup of linear operators on a normed linear space $X$ such that $\|T_s\| \leq 1$ for each $s \in S$. Let $K = \text{span}\{y - T_s y; y \in X\}$, (span (A) denotes the linear span of $A$). If $CB(S)$ has a right invariant mean then $d(x,K) = d(0,coO_x)$ for each $x \in X$.

PROOF. If $d(0,coO_x) = 0$, then since $coO_x \subseteq K - x$, $d(x,K) = 0$. If $d(0,coO_x) > 0$, then let $\phi$ be as in Theorem 3.4.1. If $z \in K$, then $\phi(z) = 0$ and $||x-z|| \geq |\phi(x-z)| = |\phi(x)| = d(0,coO_x)$.

Corollary 3.4.2 can be modified to include those semitopological semigroups for which $WAP(S)$ has a right invariant mean.

THEOREM 3.4.3. Let $S$ be a semitopological semigroup. Let $\{T_s; s \in S\}$ be a weakly continuous representation of $S$ as a semigroup of linear operators on a normed space $X$ such that $||T_s|| \leq 1$ for each $s \in S$. Suppose that $O_x$ is relatively weakly compact for each $x \in X$.

If WAP(S) has a right invariant mean, then $d(0, \mathrm{co}\,O_x) = d(x, K)$ for each $x \in X$, where $K = \mathrm{span}\,\{y - T_s y;\ y \in X,\ s \in S\}$.

PROOF. The function $\psi_x$ in the proof of Theorem 3.4.1. will be weakly almost periodic under the above conditions (Lemma 3.2.6). Applying the right invariant mean to $\psi_x$ the proof carries through as before.

COROLLARY 3.4.5. Let $G$ be a group and a semitopological semigroup. Let $\{T_g;\ g \in G\}$ be a weakly continuous representation of $G$ as a group of linear operators on a reflexive Banach space $X$ such that $\| T_g \| \leq 1$ for each $g \in G$. Then for every $x \in X$, $d(0, \mathrm{co}\,O_x) = d(x, K)$, where $K = \mathrm{span}\,\{y - T_g y;\ y \in X,\ g \in G\}$.

COROLLARY 3.4.6. Let $G$ be a locally compact group and let $\{U_g;\ g \in G\}$ be a weakly continuous, unitary representation of $G$ as a group of linear operators on a Hilbert space $\mathcal{H}$. Then $d(0, \mathrm{co}\,O_x) = d(x, K)$ for each $x \in \mathcal{H}$.

## 3.5 Examples

We illustrate some of the results of the chapter with the following examples:

EXAMPLE 3.5.1. Let $G = \{I, T, T^2, T^3\}$, where $T$ is the linear operator on $\mathbb{R}^3$ which rotates a point through $90°$ about the z-axis.

Let $x = (1,0,1)$. Then $O_x = \{(1,0,1), (0,1,1), (-1,0,1), (0,-1,1)\}$ and $\mathrm{co}\,O_x$ is the diamond, with sides of length $\sqrt{2}$, centered at $(0,0,1)$ and parallel to the x,y-plane. Note, $d(0, \mathrm{co}\,O_x) = 1$.

Define $\phi \in \mathbb{R}^{3^*}$ by $\phi(y) = x_3$ for each $y = (x_1,x_2,x_3) \in \mathbb{R}^3$. Then $\phi(x) = \|\phi\|$ and for each $y \in X$, $k = 0,1,2,3$, $\phi(T^k y) = \phi(y)$. If $G$ is given the discrete topology then $WAP(G) = \ell_\infty(G)$ and since $G$ is a group, $\ell_\infty(G)$ has a right invariant mean.

In general, if $S$ is a semitopological semigroup for which $CB(S)$ has a right invariant mean and if $\{T_s;\ s \in S\}$ is a representation of $S$ as a semigroup of linear operators on a normed linear space $X$ such that $\phi_x(s) = \phi(T_s x)$ is in $CB(S)$ for each $\phi \in X^*$, $x \in X$, then there exists a non-zero linear functional $\phi \in X^*$ for which $\Phi(T_s y) = \Phi(y)$ for each $y \in X$, $s \in S$ if and only if $d(0, \mathrm{co}\, O_x) > 0$ for some $x \in X$. Furthermore, if $d(0, \mathrm{co}\, O_x) > 0$, then we may assume that $\phi(x) = \|x\|$.

This remark follows immediately from the proof of Theorem 3.4.3. A similar result holds when $CB(S)$ is replaced by $WAP(S)$ and conditions analogous to those of Theorem 3.4.3 are imposed.

EXAMPLE 3.5.2. Let $G = SL(2,\mathbb{R})$ with the usual topology. Recall $AP(G)$ consists of constant functions only. Let $\{U_g;\ g \in G\}$ be any continuous, unitary representation of $G$ as a group of linear operators on a Hilbert space $\mathcal{H}$. Let $K$ be a compact subset of $\mathcal{H}$ which is invariant under $\{U_g;\ g \in G\}$. Let $I$ be the identity of $G$. Then, by Theorem 3.3.10, $U_g(x) = U_I(x)$ for each $g \in G$, $x \in X$.

EXAMPLE 3.5.3. Let $G$ be a group of uniformly bounded linear operators acting on the right on $\mathbb{R}^k$. Suppose that for each non-zero $x \in \mathbb{R}^k$, there exists $T \in G$ such that $Tx \neq x$. The closure of $\mathrm{co}\, O_x$ is a compact, convex subset of $\mathbb{R}^k$ which is invariant under the action of $G$. Since $G$ is a group, $AP(G)$ has a right invariant

mean. A result of Lau in [20] shows that $G$ must have a fixed point in the closure of $\operatorname{co} O_x$ (see Theorem 4.2.3). Therefore, $0$, which must be the fixed point, is in the convex hull of $O_x$. In fact, if $G$ maps any compact convex subset $K$ of $\mathbb{R}^k$ onto $K$, then $0 \in K$ and $0$ is the unique fixed point of $G$ in $K$.

# CHAPTER 4

# INVARIANT MEANS AND IDEALS

## 4.1 Introduction

A right ideal I, of a semigroup S, is a subset of S for which $IS \subseteq I$. In this chapter, we examine the role played by right ideals in determining whether or not a subspace of CB(S) will admit a right invariant mean.

We begin with a summary of a number of important fixed point theorems in the theory of invariant means (Theorem 4.2.3). We use these fixed point theorems to show that if $F$ is one of CB, WRUC, RUC, WAP or AP, and if I is a right ideal of a semitopological semigroup S, then $F(S)$ has a right invariant mean whenever $F(I)$ has a right invariant mean. The remainder of Section 2 is devoted to establishing a series of partial converses to this result. In particular, in Theorem 4.2.6 we show that CB(S) has a right invarinat mean if and only if CB(I) also has a right invariant mean.

In Section 3, we investigate implications of the existence of right thick subsets of a semigroup S. Lemma 4.3.4 demonstrates conditions under which a semitopological semigroup will contain a finite ideal which is also a group. This result is used to establish a class of semitopological semigroups S for which the existence of a right invariant mean on AP(S) is equivalent to the existence of a right invariant mean on $\ell_\infty(S)$.

The fourth section deals mainly with the existence of invariant means when S is a directed union of its subsemigroups. The results of this section are very much dependent on the criterion of Dixmier

(see Remark 2.1.3).

The final section contains examples illustrating the concepts developed earlier in the chapter. Example 4.5.1 shows that the structure of finite ideals has a great deal to do with the existence of invariant means. In fact, every semitopological semigroup $S$ can be imbedded in a semitopological semigroup $S'$ which contains at most two elements more than $S$ and $CB(S')$ has a left invariant mean while $AP(S')$ does not have a right invariant mean.

## 4.2 Fixed Points and Ideals

DEFINITION 4.2.1. A subsemigroup $I$ of a semigroup $S$ is called a *right* [resp. *left*] *ideal* of $S$ if $xs \in I$ [resp. $sx \in I$] for each $s \in S$, $x \in I$. $I$ is an *ideal* if it is both a left and right ideal.

If $S$ is a semitopological semigroup and $I$ is a right ideal of $S$, then throughout this chapter we will consider $I$ to be a semitopological semigroup with the subspace topology from $S$.

DEFINITION 4.2.2. Let $S$ be a semitopological semigroup which acts on the right on a compact, convex subset $K$ of a locally convex space $X$. The action of $S$ on $K$ is said to be *slightly continuous* if, the map $s \to ys$ is continuous from $S$ into $K$ for some $y \in K$. The action is called *jointly continuous* if, the map $(x,s) \to xs$ is continuous from $K \times S$ into $K$.

Let $E(S,K)$ denote the closure in the product space $K^K$ of the set $\{\pi_s;\ s \in S\}$, where $(x)\pi_s = xs$ for each $x \in K$, $s \in S$. The action of $S$ on $K$ is said to be *quasi-equicontinuous* if each element of $E(S,K)$ is continuous.

The next theorem is a summary of a number of important fixed point theorems needed later.

THEOREM 4.2.3. Let $S$ be a semitopological semigroup. Then

(i) [6 and 7] $CB(S)$ has a right invariant mean if and only if every slightly continuous affine right action of $S$ on a compact, convex subset $K$ of a locally convex space $X$ has a fixed point.

(ii) [24] $WRUC(S)$ has a right invariant mean if and only if every separately continuous affine right action of $S$ on a compact convex subset $K$ of a locally convex space $X$ has a fixed point.

(iii) [24] $RUC(S)$ has a right invariant mean if and only if every jointly continuous affine right action of $S$ on a compact convex subset $K$ of a locally convex space $X$ has a fixed point.

(iv) [21] $WAP(S)$ has a right invariant mean if and only if every separately continuous and quasi-equicontinuous right action of $S$ on a compact convex subset of a locally convex space $X$ has a fixed point.

(v) [20] $AP(S)$ has a right invariant mean if and only if every separately continuous and equicontinuous affine right action of $S$ on a compact convex subset $K$ of a locally convex space $X$ has a fixed point.

As a consequence of the fixed point theorems we have the following result:

THEOREM 4.2.4. Let $S$ be a semitopological semigroup and let $I$ be a right ideal of $S$. Then

(i) If $CB(I)$ has a right invariant mean then $CB(S)$ has a right

invariant mean.

(ii) If WRUC(I) has a right invariant mean then WRUC(S) has a right invariant mean.

(iii) If RUC(I) has a right invariant mean then RUC(S) has a right invariant mean.

(iv) If WAP(I) has a right invariant mean then WAP(S) has a right invariant mean.

(v) If AP(I) has a right invariant mean then AP(S) has a right invariant mean.

PROOF.

(i) Let the right action of $S$ on the compact, convex subset $K$ of a locally convex space $X$ be both affine and slightly continuous. The right action of $I$ on $K$, obtained by restricting the action of $S$ to $I$, is also affine and slightly continuous. If CB(I) has a right invariant mean then by Theorem 4.2.3. (i), the action of $I$ on $K$ has a fixed point $x_0 \in K$.

Fix $t_0 \in I$. Then $x_0 t_0 = x_0$, since $x_0$ is a fixed point. Let $s \in S$. Since $I$ is a right ideal, $t_0 s \in I$. Therefore $x_0 s = (x_0 t_0)s = x_0(t_0 s) = x_0$, and $x_0$ is a fixed point for the action of $S$. By Theorem 4.2.3 (i), CB(S) has a right invariant mean.

Straight forward modifications of the above argument will establish cases (ii), (iii) and (v).

(iv) Suppose that the affine right action of $S$ on a compact, convex subset $K$ of a locally convex space $X$ is both separately continuous and quasi-equicontinuous. If we consider the restriction of the action of $S$ to $I$, then $E(I,K) \subseteq E(S,K) \subseteq K^K$. However, each

element of $E(I,K)$ must also be continuous. The restriction of the action to $I$ is both separately continuous and quasi-equicontinuous. The result follows as before.

REMARK 4.25. For discrete semigroups the converse of Theorem 4.2.4 for $\ell_\infty$ is well known (see [23]). For general semitopological semigroups, it is not known whether the converses of the results in Theorem 4.2.4 are valid. We will show that in some instances these do indeed hold.

THEOREM 4.2.6. Let $S$ be a semitopological semigroup and let $I$ be a right ideal of $S$. Then $CB(S)$ has a right invariant mean if and only if $CB(I)$ has a right invariant mean.

PROOF. If $CB(I)$ has a right invariant mean, then $CB(S)$ also has a right invariant mean, by Theorem 4.2.4 (i).

Conversely, assume that $CB(S)$ has a right invariant mean. Then for each $s \in S$, the function $R_s$ defined by $(R_s f)(t) = f(ts)$ for each $t \in I$, $f \in CB(F)$, is a bounded linear operator on $CB(I)$. Let $S$ act on the weak-$*$ compact, convex set $K$ of means on $CB(I)$, by $\mu s = R_s^*(\mu)$ for each $\mu \in K$.

Let $t_0 \in I$. Let $\{s_\alpha\}$ be a net in $S$ which converges to $s_0 \in S$. Then $\lim_\alpha (R_{s_\alpha}^* p_{t_0})(f) = \lim_\alpha f(t_0 s_\alpha) = f(t_0 s_0) = (R_{s_0}^* p_{t_0})(f)$, for each $f \in CB(S)$. Therefore, the action of $S$ on $K$ is slightly continuous and by Theorem 2.4 (i), the action of $S$ has a fixed point $\mu \in K$. However, $R_t^* \mu = \mu$ for each $t \in I$, so $\mu$ is a right invariant mean on $CB(I)$.

DEFINITION 4.2.7. An element $e \in S$ is called a *right identity* for

$S$ if $te = t$ for each $t \in S$. A net $\{e_\alpha\}_{\alpha\in A}$ in $S$ is called an *approximate right identity* for $S$, if $\lim_\alpha te_\alpha = t$ for each $t \in S$.

THEOREM 4.2.8. Let $S$ be a semitopological semigroup and let $I$ be a right ideal of $S$. Suppose that $\{e_\alpha\}$ is an approximate right identity for $I$.

(i) $AP(S)$ has a right invariant mean if and only if $AP(I)$ has a right invariant mean.

(ii) $WAP(S)$ has a right invariant mean if and only if $WAP(I)$ has a right invariant mean.

PROOF.

(i) If $AP(I)$ has a right invariant mean, then Theorem 4.2.4 (v) implies that $AP(S)$ also has a right invariant mean.

Conversely, assume that $AP(S)$ has a right invariant mean. Define $(R_s f)(t) = f(ts)$ for each $f \in AP(I)$, $t \in I$, and $s \in S$. Since $R_s$ is linear and norm decreasing, it follows that $R_s f \in AP(I)$. Hence, $R_s$ is an operator on $AP(I)$ into $AP(I)$. Following an idea of Lau in [20, Theorem 3.2], we define for each $f \in AP(I)$ the pseudonorm $q_f$ on $AP(I)^*$ by

$$q_f(\phi) = \sup\{|\phi(R_t f)|, |\phi(f)|;\ t \in I\}.$$

Since $\{e_\alpha\}$ is an approximate right identity $R_{e_\alpha s} f$ converges in norm to $R_s f$ for each $f \in AP(I)$, $s \in S$.

Hence

$$q_f(\phi) = \sup\{|\phi(R_s f)|, |\phi(f)|; s \in S\}.$$

Let $Q = \{q_f; f \in AP(I)\}$. On the weak-* compact, convex set $K$

of means on $AP(I)$, the weak-$*$ topology agrees with the topology of uniform convergence on totally bounded subsets of $AP(I)$. Therefore $Q$ determines the weak-$*$ topology on $K$.

Let $S$ act on $K$ by $\mu s = R_s^*\mu$ for each $\mu \in K$, $s \in S$. The action of $S$ on $K$ is affine and $Q$-non-expansive. Therefore, it is equicontinuous with respect to the weak-$*$ topology on $K$.

Now suppose $\{s_\gamma\}$ is a net in $S$ which converges to $s \in S$. Then $\lim_\gamma(\lim_\alpha R_{e_\alpha s_\gamma} f(t)) = f(ts) = \lim_\alpha(\lim_\gamma R_{e_\alpha s_\gamma} f)(t)$ for each $t \in I$. It follows that $R_{s_\alpha} f$ converges in norm to $R_s f$ for each $f \in AP(I)$. Therefore, the action of $S$ on $K$ is also separately continuous. By Theorem 4.2.3 (v), the action of $S$ has a fixed point $\mu \in K$, which is a right invariant mean on $AP(I)$.

(ii) If $WAP(I)$ has a right invariant mean, then, by Theorem 4.2.4 (iv), $WAP(S)$ also has a right invariant mean.

Assume that $WAP(S)$ has a right invariant mean.

Define $R_s f(t) = f(ts)$ for each $f \in WAP(I)$, $t \in I$, $s \in S$. Since $O_R(f)$ is relatively weakly compact, $R_s f$ is the weak limit of $R_{e_\alpha s} f$ for each $f \in WAP(I)$. Therefore, $R_s$ is a bounded linear operator on $WAP(I)$. Similarly, if $\{s_\gamma\}$ is a net in $S$ which converges to $s \in S$, then $R_{s_\alpha} f$ converges to $R_s f$ in the weak topology of $CB(I)$, for each $f \in WAP(I)$.

Let $\mathcal{T}$ be the topology on $WAP(I)^*$ determined by the family of pseudonorms $Q = \{q_f;\ f \in WAP(I)\}$, where each $q_f$ is defined as in part (i). By the Mackey-Arens Theorem, the dual of $WAP(I)^*$ with respect to the topology $\mathcal{T}$ is $WAP(I)$.

The right linear action of $S$ on the set $K$, of means on $WAP(I)$, defined by $\mu s = R_s^*\mu$ is both separately continuous and quasi-

equicontinuous with respect to the topology $\mathcal{T}$. But, since K is convex and $\mathcal{T}$-compact, the action must, by Theorem 4.2.3 (iv), have a fixed point $\mu \in K$. It follows that $\mu$ is a right invariant mean on WAP(I).

THEOREM 4.2.9. Let S be a semitopological semigroup and let I be a right ideal of S. Let e be a right identity for I. Then

(i) WRUC(S) has a right invariant mean if and only if WRUC(I) has a right invariant mean.

(ii) RUC(S) has a right invariant mean if and only if RUC(I) has a right invariant mean.

PROOF.

(i) If WRUC(I) has a right invariant mean, then by Theorem 4.2.4 (ii), WRUC(S) also has a right invariant mean.

Assume that WRUC(S) has a right invariant mean. Define $(R_s f)(t) = f(ts)$ for each $f \in$ WRUC(I), $t \in I$, $s \in S$. Since $R_s$ is continuous when CB(I) is given the pointwise topology, $R_s$ is an operator from WRUC(I) into WRUC(I) (see [2, p. 101]).

Let S act on the weak-$*$ compact, convex set K of means on WRUC(I) by $\mu s = R_s^* \mu$ for each $\mu \in K$, $s \in S$. Let $\{s_\alpha\}$ be a net in S which converges to s. Let $f \in$ WRUC(I) and $\mu \in K$. Then $\lim_\alpha (R_{s_\alpha}^* \mu)(f) = \lim_\alpha \mu(R_{s_\alpha} f) = \lim_\alpha (R_{es_\alpha} f) = \mu(R_{es} f) = \mu(R_s f) = (R_s^* \mu)(f)$. Hence the right affine action of S on K is separately continuous. By Theorem 4.2.3 (ii), the action of S on K has a fixed point $\mu$. Then $\mu$ is a right invariant mean on WRUC(I).

(ii) If RUC(I) has a right invariant mean, then by Theorem 4.2.4

(iii), RUC(S) also has a right invariant mean.

Assume that RUC(I) has a right invariant mean. $R_s$ defined by $(R_s f)(t) = f(ts)$, is a bounded linear operator from RUC(I) into RUC(I). To see this observe that if $\{t_\alpha\}$ is a net in I which converges to t then $t_\alpha s$ converges to ts and hence $\lim_\alpha R_{t_\alpha}(R_s f) = \lim_\alpha(R_{t_\alpha s} f) = (R_{ts} f) = R_t(R_s f)$ in the norm topology of CB(S).

Let S act on the weak-$*$ compact, convex set K of means on RUC(I) by $\mu s = R_s^* \mu$ for each $\mu \in K$, $s \in S$. Let $\mu \in K$, $s \in S$ and let $\{\mu_\alpha\}$, $\{s_\alpha\}$ be nets such that $\mu_\alpha \to \mu$ in the weak-$*$ topology and $s_\alpha \to s$. Let $f \in RUC(S)$. Since e is a right identity for I we have

$$\begin{aligned}0 &\leq \lim_{\gamma,\alpha} |(R^*_{s_\gamma} \mu_\alpha)(f) - (R^*_s \mu)(f)| \\ &\leq \lim_{\gamma,\alpha} |(\mu_\alpha(R_{s_\gamma} f - R_s f) + ((\mu_\alpha - \mu)(R_s f))| \\ &\leq \lim_\gamma \|R_{s_\gamma} f - R_s f\| + \lim_\alpha |(\mu_\alpha - \mu)(R_s f)| \\ &= \lim_\gamma \|R_{es_\gamma} f - R_{es} f\| + \lim_\alpha |(\mu_\alpha - \mu)(R_s f)| \\ &= 0\end{aligned}$$

Therefore the action of S on K is jointly continuous and affine. By Theorem 4.2.3 (iii), the action has a fixed point $\mu \in K$ which is a right invariant mean on RUC(S).

It is well known that for a locally compact group G, CB(G) has a right invariant mean if and only if RUC(G) has a right invariant mean (see [13, Theorem 2.2.1]. This gives us the following corollary to Theorem 4.2.9.

COROLLARY 4.2.10. Let S be a semitopological semigroup and let G

be a locally compact group and a right ideal of $S$. Then $CB(S)$ has a right invariant mean if and only if $RUC(S)$ has a right invariant mean.

PROOF. If $\mu$ is a right invariant mean on $CB(S)$, then the restriction of $\mu$ to $RUC(S)$ is a right invariant mean on $RUC(S)$.

If $RUC(S)$ has a right invariant mean, then Theorem 4.2.9 implies that the locally compact group $G$ is such that $RUC(G)$ has a right invariant mean also. The previous remark states that if $RUC(G)$ has a right invariant mean then so must $CB(G)$. However, for the right ideal $G$, $CB(G)$ has a right invariant mean implies that $CB(S)$ has a right invariant mean (Theorem 4.2.6).

THEOREM 4.2.11. Let $S$ be a semitopological semigroup and let $I$ be a finite right ideal of $S$. Then $AP(S)$ has a right invariant mean if and only if $\ell_\infty(S)$ has a right invariant mean.

PROOF. The restriction of any right invariant mean $\mu$ on $\ell_\infty(S)$ to $AP(S)$ is a right invariant mean on $AP(S)$.

Conversely, suppose that $AP(S)$ has a right invariant mean. Let $K \subseteq \ell_\infty(I)^*$ be the set of means on $\ell_\infty(I)$. Since $I$ is finite, $\ell_\infty(I)^*$ is finite dimensional and the weak-$*$ and norm topologies agree.

Let $S$ act on the norm compact, convex set $K$ by $\mu s = R_s^*\mu$ for each $\mu \in K, s \in S$. The action of $S$ on $K$ is affine and separately continuous with respect to the norm topology on $K$. By Theorem 4.2.3 (v), and the assumption that $AP(S)$ has a right invariant mean, then action of $S$ has a fixed point $\mu_0 \in K$. Thus $\mu_0$ is a right invariant mean on $\ell_\infty(I)$. (Apply Theorem 4.2.9).

COROLLARY 4.2.12. Let $S$ be a semitopological semigroup and let $I_0$ be a finite right ideal of $S$. Then the following are equivalent

(i) $AP(I)$ has a right invariant mean for some right ideal $I$ of $S$.

(ii) $\ell_\infty(I)$ has a right invariant mean for any right ideal $I$ of $S$.

PROOF. The proof is a straight forward application of Theorems 4.2.11 and 4.2.6.

REMARK 4.2.13. Let $G$ be any group and let $I$ be any finite semigroup for which $\ell_\infty(I)$ does not have a right invariant mean. Corollary 4.2.12 implies that there does not exist any semigroup $S$ for which $G$ and $I$ are both right ideals.

## 4.3 Invariant Means and Right Thick Subsets

The purpose of this section is to establish the equivalence of the existence of a right invariant mean on $\ell_\infty(S)$ with the existence of a right invariant mean on $WAP(S)$ for a class of semitopological semigroups which may not contain finite right ideals.

DEFINITION 4.3.1. A subset $F$ of a semigroup $S$ is said to be *right thick* in $S$ if for any finite subset $A$ of $S$ there exists an element $x \in S$ such that $xA = \{xa; a \in A\} \subseteq F$.

REMARK 4.3.2. The above definition is due to Mitchell (see [23]). It is clear that any right ideal of $S$ must also be right thick. In addition, if $S$ is a semigroup for which $\ell_\infty(S)$ has a left invariant

mean then it is not difficult to show that any left ideal of $S$ is right thick in $S$ (see [23, p. 257]).

We will need the following generalization of a result of Mitchell [23, p. 257].

LEMMA 4.3.3 [Mitchell]. Let $S$ be a semitopological semigroup. Let $\mathcal{F}$ be a right introverted, norm closed, conjugate closed subspace of $CB(S)$ which contains the constants. Suppose that $\mathcal{F}$ has a right invariant mean $\mu$. If $F$ is a right thick subset of $S$, then there is a right invariant mean $\mu$ on $\mathcal{F}$ such that $\mu(X_F) = 1$.

PROOF. Since $F$ is right thick there exists a multiplicative mean $\nu$ on $\ell_\infty(S)$ such that $\nu(R_s X_F) = 1$ for each $s \in S$.

For each $f \in \mathcal{F}$, define $\nu_f(s) = \nu(R_s f)$. Since $\mathcal{F}$ is right introverted, $\nu_f \in \mathcal{F}$. Let $\phi(f) = \mu(\nu_f)$. Then $\phi$ is a right invariant mean on $\mathcal{F}$ and $\phi(X_F) = 1$

LEMMA 4.3.4. Let $S$ be a semitopological semigroup. Let $\mathcal{F}$ be a right introverted, norm closed, conjugate closed subspace of $CB(S)$ which contains $AP(S)$. Suppose that $\mathcal{F}$ has a right invariant mean. If $S$ contains a finite right thick subset $F$ such that $X_F \in \mathcal{F}$, then $S$ contains a right ideal $G$ which is a group.

PROOF. $F$ is right thick, so by Lemma 4.3.3, there exists a right invariant mean $\mu$ on $\mathcal{F}$ such that $\mu(X_F) = 1$. Since $F$ is finite there exists $a \in F$ such that $\mu(X_{\{a\}}) > 0$.

Let $n \in N$ be such that $n\mu(X_{\{a\}}) > 1$. Suppose that the set $aS$ contains $n$ distinct elements $\{as_1,\ldots,as_n\}$. Then $\mu(X_{\{as_i\}}) =$

$\mu(R_{s_i}X_{\{as_i\}}) \geq \mu(X_{\{a\}})$ implies that $\mu(X_{\{as_1,\ldots,as_n\}}) \geq n\mu(X_{\{a\}}) > 1$, which is impossible. Therefore $aS$ is a finite right ideal of $S$.

Let $t \in S$ be such that the cardinality of $tS$ is minimal. Let $G = tS$. Since the cardinality of $tS$ is minimal, $G$ is left cancellative. (That is, if $xy = xz$, $x, y, z \in G$, then $y = z$). $AP(S)$ has a right invariant mean implies that the finite right ideal $G$ is such that $AP(G)$ has a right invariant mean (Corollary 3.2.12). However, for a finite semigroup $G$, $AP(G) = \ell_\infty(G)$.

A finite, left cancellative semigroup for which $\ell_\infty$ has a right invariant mean, must be a group (see [27, pp. 1078, 1079]).

THEOREM 4.3.5. Let $S$ be a semitopological semigroup. Let $F$ be a finite, right thick subset of $S$ such that $X_F \in WAP(S)$. Then $\ell_\infty(S)$ has a right invariant mean if and only if $WAP(S)$ has a right invariant mean.

PROOF. If $WAP(S)$ has a right invariant mean, then Lemma 4.3.4 implies that $S$ contains a group $G$ which is also a right ideal. $G$ is a finite group so $\ell_\infty(G)$ has a right invariant mean. By Theorem 4.2.6, $\ell_\infty(S)$ also has a right invariant mean.

REMARK 4.3.6. It is not known whether there exists a semitopological semigroup $S$ for which $AP(S)$ has a right invariant mean but $WAP(S)$ does not.

A topological semigroup $S$, with a finite right thick subset $F$ for which $X_F \in WAP(S)$, $AP(S)$ has a right invariant mean but $CB(S)$ does not, would provide such an example.

## 4.4. Directed Unions of Semigroups

DEFINITION 4.4.1. Let $\{S_\alpha\}$ be a family of semigroups. A semigroup $S$ is said to be a *directed union* of the family $\{S_\alpha\}$ if $S_\alpha$ is a subsemigroup of $S$, for each $\alpha \in A$, $S = \bigcup_{\alpha\in A} S_\alpha$, and for each $\alpha$, $\beta \in A$ there exists $\gamma \in A$ such that $S_\alpha \subset S_\gamma$ and $S_\beta \subset S_\gamma$.

LEMMA 4.4.2. Let $S$ be a semitopological semigroup which is the directed union of a family $\{S_\alpha\}_{\alpha\ A}$ of subsemigroups of $S$. Let each $S_\alpha$ have the subspace topology inherited from $S$. Let $F$ be a right invariant, norm closed, conjugate closed subspace of $CB(S)$. For each $\alpha \in A$, let $F_\alpha$ denote the subspace of $CB(S_\alpha)$ consisting of the restrictions of each element of $F$ to $T_\alpha$. If $F_\alpha$ has a right invariant mean for each $\alpha \in A$, then $F$ has a right invariant mean.

PROOF. The result follows immediately from Dixmier's criterion.

LEMMA 4.4.3. Let $S$ be a semitopological semigroup and let $T$ be a subsemigroup of $S$ with the subspace topology. Let $F$ be one of $CB(S)$, $WRUC(S)$, $RUC(S)$, $WAP(S)$ or $AP(S)$. Then if $f \in F$, the restriction $f_{1_T}$ of $f$ to $T$ is in $CB(T)$, $WRUC(T)$, $RUC(T)$, $WAP(T)$, or $AP(T)$ respectively.

The proof of this lemma is straight forward and will be omitted.

THEOREM 4.4.4. Let $S$ be a semitopological semigroup which is the directed union of a family $\{S_\alpha\}_{\alpha\in A}$ of semigroups. Then

(i) If $CB(S_\alpha)$ has a right invariant mean for each $\alpha \in A$, then

$CB(S)$ has a right invariant mean.

(ii) If $WRUC(S_\alpha)$ has a right invariant mean for each $\alpha \in A$, then $WRUC(S)$ has a right invariant mean.

(iii) If $RUC(S_\alpha)$ has a right invariant mean for each $\alpha \in A$, then $RUC(S)$ has a right invariant mean.

(iv) If $WAP(S_\alpha)$ has a right invariant mean for each $\alpha \in A$, then $WAP(S)$ has a right invariant mean.

(v) If $AP(S_\alpha)$ has a right invariant mean for each $\alpha \in A$, then $AP(S)$ has a right invariant mean.

PROOF. Apply Lemma's 4.4.2 and 4.4.3.

## 4.5 Examples

The first example illustrates the important role played by finite right ideals in determining the existence of right invariant means.

EXAMPLE 4.5.1. Let $(S,\cdot)$ be any semitopological semigroup and let $S' = S \cup \{A,B\}$. Define a multiplication "$\circ$" on $S'$ by

(i) $s\circ t = st$ for each $s, t \in S$

(ii) $s\circ A = A\circ s = A$, $s\circ B = B\circ s = B$

(iii) $A\circ B = B\circ B = B$, $B\circ A = A\circ A = A$.

It is easy to check that the multiplication on $S$ is associative and hence that $S$ is a semigroup.

Let $\mathcal{B}$ be a basis for the topology of $S$. Let $\mathcal{B}' = \mathcal{B} \cup \{A\} \cup \{B\}$ be a basis for a topology $\mathcal{T}'$ on $S'$. Then $(S',\circ)$ together with $\mathcal{T}'$ is a semitopological semigroup with $S$ as a subsemigroup.

Let $I = \{A,B\}$. Then $I$ is an ideal of $(S',\circ)$. Also, $\ell_\infty(I)$ has a left invariant mean but not a right invariant mean. Theorems 4.2.6 and 4.2.13, imply that $CB(S')$ has a left invariant mean while $AP(S')$ does not have a right invariant mean. We have, in fact, shown the following:

Any semitopological semigroup $S$ is such that there exist a semitopological semigroup $S'$ satisfying

(i) $S \subset S'$, $S'$ contains at most 2 elements more than $S$.

(ii) $CB(S')$ has a left invariant mean but $AP(S')$ does not have a right invariant mean.

EXAMPLE 4.5.2. An element $s$ in a semigroup $S$ is called a *left zero* of $S$ if $st = s$ for each $t$, $s \in S$.

Let $S = Y \times T$ be a semitopological semigroup such that $Y$ and $T$ are semigroups, and $Y$ contains a left zero $y_0$. Then, for each $(y,t) \in S$, and $\{y_0\} \times T$, $(y_0,s)(y,t) = (y_0 y, st) = (y_0, st)$. Thus $\{y_0\} \times T$ is a right ideal of $S$. If $WAP(T)$ [resp. $AP(T)$] has a right invariant mean, then so does $WAP(S)$ [resp. $AP(S)$] (Theorem 4.2.3).

A semigroup $S$ of the form $S = Y \times G$, where $G$ is a group and $Y$ consists of left zeros only, is called a *left group*. From the above remark it follows that if $S$ is a left group and a semitopological semigroup, then $WAP(S)$ has a right invariant mean. If $S$ is a compact left group, then $CB(S)$ has a right invariant mean.

EXAMPLE 4.5.3. Let $S$ be a compact semitopological semigroup and let "e" be a primitive idempotent of $S$ (see [2, p. 31]). Then

Se is a left group and a left ideal of S. Therefore, WAP(Se) has a right invariant mean. Thus every compact semitopological semigroup has a left ideal I for which WAP(I) has a right invariant mean even if WAP(S) does not.

EXAMPLE 4.5.4. A semigroup S is called *periodic* if the subsemigroup generated by each element of S is finite. A right cancellative periodic semigroup is a left group (see [22, pp. 239, 245]).

Let S be a right cancellative semitopological semigroup. Let T be a periodic subsemigroup of S. Then WAP(T) has a right invariant mean. Furthermore, if T is a right ideal of S then WAP(S) has a right invariant mean.

EXAMPLE 4.5.5. Let $S = \{A,B\}$ with multiplication defined as in Example 4.5.1. Then the 1-dimensional space $F$ consisting of constant functions only, is maximal in the set of subspaces of $\ell_\infty(S)$ which do not contain $AP(S) = \ell_\infty(S)$. The set S is finite, right thick in S, and $X_S \in F$. However the space $F$ has a right invariant mean while S does not contain any right ideal which is a group. This shows that the condition $AP(S) \subseteq F$, can not be dropped in the statement of Lemma 4.3.4.

EXAMPLE 4.5.6. The condition, in Lemma 4.3.4. that F be finite can not be omitted. Let $S = \{1,2,\ldots\}$ with usual addition and the discrete topology. Since S is commutative, $\ell_\infty(S)$ has a right invariant mean (see [16, p. 231]). However S is right thick in itself but S does not contain any right ideal which is a group.

EXAMPLE 4.5.7. Let $S_0 = \{A,B\}$ with multiplication as defined in

example 4.5.1. Let $G_0$ be the free group with two generators.

Let $G$ be any group. Since $AP(S_0)$ does not have a right invariant mean, Corollary 4.2.12 implies that no semitopological semigroup can contain both $G$ and $S_0$ as right ideals.

Let $S$ be any finite semigroup $\ell_\infty(G_0)$ does not have a right invariant mean, while $AP(G_0)$ does. Corollary 4.2.12 again implies that no semitopological semigroup exists for which $S$ and $G_0$ are both right ideals.

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